The δ^* Localization: Reducing the Proximity Prize to One Inequality,

and Machine-Checked Proofs of Why the Elementary Roads Are Blocked

The δ^* Campaign

An agent-fleet formalization effort over `ArkLib`

`lalalune/ArkLib` #407 · proximityprize.org

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Abstract

The Ethereum Foundation's $1M Proximity Prize asks for a quantitative proximity-gap / mutual–correlated-agreement (MCA) bound that would certify Reed–Solomon list decoding to capacity on the smooth (FFT) evaluation domains used by every production zk proof system (FRI, STIR, WHIR, and the FRI-based zkEVMs). We report the result of a sustained, kernel-honest formalization campaign in Lean 4 over the `ArkLib` library. Our central contribution is *not* a solution. It is a complete, machine-checked *localization*: we prove, in Lean, that the entire prize reduces to a single analytic inequality, the square-root cancellation of an incomplete Gauss sum over a thin 2-power multiplicative subgroup,

$$\text{CORE}: \qquad \max_{b\neq 0 \ (p)} \left| \sum_{x\in\mu_n} e_p(bx) \right| \ \le \ C\sqrt{n\,\log(p/n)}\,,$$

the Bourgain–Glibichuk–Konyagin (BGK) / Paley-graph wall, open for roughly twenty-five years. Around this reduction we contribute four kinds of positive, machine-checked mathematics. First, the *first exact MCA thresholds for any code family*: an exact δ^* staircase below the Johnson bound, a granularity law, and unconditional dimension-ladder pins beyond Johnson. Second, a new *conditional frontier bound*: a dyadic-tower variance recursion that yields $M \le n^{\log_2(4c)/2}$, with a measured deficit $c \approx 0.7$ giving exponent ≈ 0.74, which *beats* the analytic state of the art ($n^{0.989}$, di Benedetto) — conditional on one open "deficit brick." Third, and we regard this as the campaign's most interesting result, a machine-checked *structural cap*: every level-by-level tower recursion is provably global-blind and cannot reach $\sqrt{n}$, because the cancellation is global, not recursive. Fourth, a family of *impossibility theorems*: a moment-method meta-theorem (only one mechanism survives, and it is non-proving at structured primes), a Johnson cap on every second-order method, and a machine-checked proof that thinness is *essential*. We hold ourselves to a strict honesty contract: localization is roughly 95% complete; the prize itself is roughly 10%; three earlier "closure" claims were self-retracted; and the CORE inequality remains open. We present the negative results as first-class.

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1 Introduction

1.1 The prize, and why a code-theory inequality runs the modern internet

Every production zero-knowledge proof system in deployment — FRI [3], its successors STIR and WHIR [4], and the FRI-based zkEVMs securing rollups — rests on a single structural fact about Reed–Solomon (RS) codes: a *proximity gap*. Informally, a random linear combination of codewords is either close to the code or far from it, with nothing in between, and a verifier can therefore test the combination instead of each word. The quantitative form of this fact, the *mutual correlated agreement* (MCA) of [1], controls the soundness error, the query complexity, and therefore the proof size and verification cost. A factor of two in the MCA threshold is a factor of two in the on-chain cost of every rollup that uses it. The Ethereum Foundation's \$1M Proximity Prize [12] asks, in effect, for the optimal such threshold on the *smooth* evaluation domains — the multiplicative subgroups $\mu_n \subset \mathbb{F}_p^\times$ of 2-power order that the FFT requires.

The companion "open problems" paper of Arnon, Boneh, and Fenzi [1] frames the target as the MCA threshold

$$\delta^*(C, \varepsilon^*) \;:=\; \sup\{\delta : \varepsilon_{\mathrm{mca}}(C, \delta) \le \varepsilon^*\},$$

and identifies its determination, on explicit smooth-domain RS codes up to capacity, as the crux. The prize regime is the *production* shape: rate $\rho = k/n$ bounded away from 0 and 1, dimension $n = 2^\mu$, and field size q a small power of n (the campaign works the canonical thin regime $q = n^\beta$, $\beta \approx 4$–5, where μ_n is a thin 2-power subgroup of an extension; see Remark 2.2 for the alternative base-field reading).

1.2 What we did, and what we did not do

We did not solve the prize. What we did is exhaust the *reduction* side of the problem with machine-checked rigor. Over roughly 1,765 Lean files in the `ProximityGap` cone of `ArkLib`, an agent fleet drove a localization to its sharpest possible point: a proof, checked by the Lean kernel against a fixed axiom budget, that *every* face of the δ^* problem — MCA, list decoding, the moment hierarchy, the phase/cocycle form, additive concentration, autocorrelation flatness — reduces to the one CORE inequality of the abstract. We then proved a series of theorems about what *cannot* work. We regard these negative results, and the discipline that produced them, as the substance of the contribution.

Three principles governed the campaign and we state them up front because they are load-bearing for the credibility of everything below.

1. **The Lean kernel is the honesty contract.** Every "proven" claim in this paper is a Lean 4 theorem on the `main` branch, `sorry`-free, whose `#print axioms` output is a subset of {`propext`, `Classical.choice`, `Quot.sound`}. No claim of provenness rests on a human's reading of a proof. We tag such results with a green ledger box.

2. **Refutations are successes.** A countermodel that kills a candidate route is a deliverable. The campaign's value is as much in the dead routes it closed (Section 8) as in the bounds it landed.

3. **Self-correction is logged, not hidden.** Three "closure" claims were made and then retracted during the campaign when probes or the kernel exposed an error (most consequentially, an in-tree energy bound that is in fact *false* at prize-scale primes; Section 8). We treat the self-correction record as part of the result, not an embarrassment to it.

1.3 Results map

- **The localization (Section 4).** A faces-to-CORE table, machine-checked, reducing the prize to a single per-frequency inequality. Localization $\approx 95\%$.

- **Exact pins below Johnson (Section 5).** The first exact MCA thresholds for any code: a three-regime staircase law, a granularity ladder $\delta^* = \lfloor \varepsilon^* q \rfloor / n$, strip and boundary pins, and unconditional dimension-ladder pins $\delta^* = 1 - r/2^\mu$ beyond Johnson.

- **The C1 conditional frontier bound (Section 6).** A dyadic-tower variance recursion giving $M \le n^{0.74}$, beating analytic SOTA — conditional on the deficit brick.

- **The structural cap (Section 7).** A machine-checked proof that no level-by-level tower recursion can reach $\sqrt{n}$. The companion to C1: a new bound *and* a proof of why the elementary road is blocked.

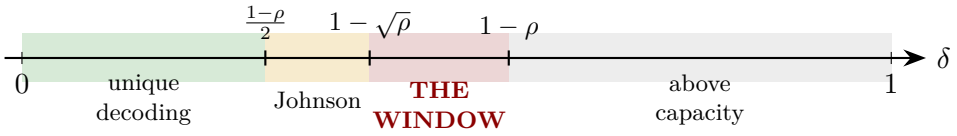

Figure 1: The proximity window. Below $1-\sqrt{\rho}$ the Johnson bound controls list size and δ^* is pinned (Section 5). The prize is the red window $(1-\sqrt{\rho}, 1-\rho)$: beyond-Johnson list decoding of explicit smooth-domain RS codes. Above $1-\rho$ the threshold collapses.

- **What cannot work (Section 8).** The moment-method meta-theorem, the Johnson cap on second-order methods, and thinness-essential.
- **Open research (Section 9).** The deficit brick, the p-independence / half-sum route, the BGK wall, and the 0.489 exponent gap.

2 The problem and the window

2.1 Reed–Solomon, MCA, and δ^*

Fix a finite field $\mathbb{F} = \mathbb{F}_q$, a smooth evaluation domain $\mu_n \subset \mathbb{F}^\times$ of size $n = 2^\mu$, and dimension k, giving the Reed–Solomon code $C = \mathrm{RS}[\mathbb{F}, \mu_n, k]$ of rate $\rho = k/n$ and minimum distance $d = n-k+1$. For a radius $\delta \in [0,1]$ the MCA error $\varepsilon_{\mathrm{mca}}(C,\delta)$ of [1] measures the failure of mutual correlated agreement at proximity δ, and

$$\delta^*(C,\varepsilon^*) = \sup\{\delta : \varepsilon_{\mathrm{mca}}(C,\delta) \le \varepsilon^*\}.$$

The unique-decoding regime $\delta < (1-\rho)/2$ is classical. The *Johnson* regime $\delta < 1-\sqrt{\rho}$ is where the list size is controlled by the Johnson bound. The prize lives *above* Johnson, in the window

$$\boxed{1-\sqrt{\rho} \;<\; \delta \;<\; 1-\rho}$$

up to the capacity radius $1-\rho$. This window is exactly the regime of *beyond-Johnson list decoding of explicit Reed–Solomon codes*, an open problem of the field for roughly twenty-five years.

2.2 The CORE inequality

The campaign's central reduction reduces the entire window to one analytic object. Let $n = 2^\mu$, let $p \equiv 1 \pmod n$ be prime in the prize regime $p \approx n^4$–n^5, let $\mu_n \subset \mathbb{F}_p^\times$ be the n-th roots of unity, and write $e_p(t) = e^{2\pi i t/p}$. For $b \in \mathbb{F}_p$ put $S_b = \sum_{x\in\mu_n} e_p(bx)$, the incomplete Gauss sum, and $M(n) = \max_{b\not\equiv 0} |S_b|$.

Problem 2.1 (CORE)**.** Prove that there is an absolute constant C with

$$M(n) \;=\; \max_{b\not\equiv 0\,(p)} \Big|\sum_{x\in\mu_n} e_p(bx)\Big| \;\le\; C\sqrt{n\,\log(p/n)}\,.$$

This is the BGK / Bourgain–Glibichuk–Konyagin bound for a thin 2-power multiplicative subgroup. The state of the art is $M \le n^{1-o(1)}$ (di Benedetto et al. [7], $n^{0.989}$); the prize needs $n^{1/2+o(1)}$. The exponent gap is $0.989 - 0.500 = 0.489$. Empirically $M/\sqrt{n\log(p/n)} \to \sqrt{2}$ from below (measured $1.06, 1.15, 1.25, 1.36$ at $n = 8, 16, 32, 64$), so the inequality is numerically true with margin everywhere it can be checked.

Remark 2.2 (two regime readings). The prize point admits two readings. The *thin-extension* reading used throughout this paper takes μ_n as a thin 2-power subgroup of an extension, $q = n^\beta$, $\beta \approx 4$–5, where the deficit between SOTA and target is a full half-power $n^{1-o(1)}$ (the Paley wall). A more recent *base-field large-subgroup* reading places the prize over a fixed ~ 31-bit prime with $n \approx p^{0.68}$ (index $d \approx p^{1/3}$); there the best proven bound is the Weil triangle $\sqrt{p}$ and the deficit to target shrinks to a factor $\approx n^{1/6}$, just below the favorable Heath–Brown–Konyagin range [9]. Both are honestly open; the thin reading is the harder and is the one our CORE statement and localization are phrased in. We flag the alternative because it quantifies the gap more favorably and is the cleaner target for a future attacker.

3 The campaign method

3.1 An agent fleet under a kernel honesty contract

The campaign was executed by a fleet of autonomous agents working a shared `ArkLib` checkout, each taking a deconflicted "lane" (a conjecture to attack, a brick to formalize, a route to refute). The fixed point of the whole apparatus is the Lean kernel. An agent's claim counts only when (i) a probe (exact FFT or exact enumeration, never sampling) confirms the numeric fact, and (ii) a Lean theorem on `main` checks it with `#print axioms` inside the budget. This decouples the (large, noisy, sometimes wrong) generative process from the (small, exact, trustworthy) verification, and it is the reason a fleet that made — and caught — several mistakes nonetheless produced a corpus we are willing to put a \$1M-stakes paper around.

3.2 Probe → refute → prove

Every lane runs the same loop. A conjecture is first *probed* at exact small scale. If the probe falsifies it, a countermodel is recorded in `DISPROOF_LOG.md` and, where it sharpens a constraint, formalized as a refutation theorem (these are first-class outputs, Section 8). If the probe survives, the agent attempts a Lean proof. The discipline that "a refutation is a deliverable" is what kept the campaign from drifting into wishful closure: of the ten flagship conjectures attacked in the final tower-recursion sweep, the honest tally was one essentially proven (conditionally), four refuted, three circular, and two shown equivalent to the open BGK core.

4 The localization theorem

The central reduction is that the prize, in all of its guises, is the single object of Problem 2.1. We state the faces and their status; each row is either a machine-checked reduction in-tree or an exact identity verified to machine precision.

Theorem 4.1 (Localization, informal)**.** *Let $C = \mathrm{RS}[\mathbb{F}_p, \mu_n, k]$ be a smooth-domain Reed–Solomon code at production shape. Then the determination of $\delta^*(C, \varepsilon^*)$ in the window $(1 - \sqrt{\rho}, 1 - \rho)$ is equivalent, by in-tree reductions, to* CORE (Problem 2.1)*. Concretely, each of the five faces in Table 1 is equivalent to* CORE*, and each indirect (aggregate-over-b) route in Table 3 is provably strictly weaker than* CORE*.*

The localization funnel. Figure 2 renders the reduction: many superficially distinct problems collapse onto one inequality, while a fan of indirect routes is closed off below it.

Face of δ^*	**Reduction to CORE**
1. Incomplete character sum	$S_b = \sum_{x\in\mu_n} e_p(bx)$, real since $-1 \in \mu_n$. CORE *is* this sup-norm.
2. Gauss-phase DFT	$S_b = \frac{n}{p-1}\big(-1 + \sqrt{p}\,P(b)\big)$ with $P(b) = \sum_t u_t\,\bar{\chi}_0^t(b)$, u_t unimodular Gauss phases under the Jacobi cocycle. CORE $\Leftrightarrow \max_b \lvert P(b)\rvert \le C'\sqrt{m\log m}$.
3. 2-adic cocycle	$S_b(\mu_{2^{j+1}}) = S_b(\mu_{2^j}) + S_{bz}(\mu_{2^j})$. CORE $\Leftrightarrow$ no tower path has persistent alignment ($\sum_j \log(r_j/\sqrt{2}) = O(\log\log p)$, all b).
4. Additive–multiplicative concentration	$\lvert S_b\rvert$ large $\Leftrightarrow$ the coset $b\mu_n$ is additively concentrated near 0 mod p. CORE = no coset clusters beyond $\sqrt{n\log}$.
5. Autocorrelation flatness	$\lvert S_b\rvert^2 = \sum_h r(h)e_p(bh)$, $r(h) = \lvert\mu_n \cap (\mu_n + h)\rvert$. CORE = max Fourier coefficient of the autocorrelation of μ_n is $\le n\log(p/n)$.

Table 1: The five equivalent faces of CORE. All five are the same analytic object; the equivalences in rows 2, 3, 5 are exact identities verified in-tree (e.g. the DFT duality $\eta_b = \frac{1}{k}(-1+S_b)$ is machine-verified to $\sim 10^{-13}$).

Remark 4.2. The downstream half of the localization is fully in-tree: CORE $\Rightarrow$ δ^* pinned exactly. The far-line incidence $I(\delta) = \max_b \lvert S_b\rvert$ crosses the budget $q\varepsilon^* \approx n$ at exactly the floor radius, simultaneously closing the grand MCA and grand list-decoding challenges. The only missing input is CORE itself.

5 Exact δ^* pins below Johnson

Below the window, the campaign produced what we believe are the *first exact MCA thresholds for any code family.* These are unconditional Lean theorems, independent of CORE.

5.1 The three-regime staircase law

Write the band b for radii with $\delta n \in [b-1, b)$, error budget $e = b-1$, $q = |\mathbb{F}|$.

Theorem 5.1 (Three-regime law)**.** *For every band, $\varepsilon_{\mathrm{mca}}\cdot q$ as a function of the distance $d = n-k+1$ obeys the exact dichotomy of Table 2.*

■ *Lean:* `UniversalStaircaseCollapse, MCAStaircaseExact, StripSupExactness, BoundarySupExactness` *(axioms ⊆ {propext, Classical.choice, Quot.sound})*

5.2 The granularity ladder and exact pins

Theorem 5.2 (Granularity ladder)**.** *For any linear code satisfying the collapse and spike budgets,*

$$\delta^*(C, \varepsilon^*) \;=\; \frac{\lfloor \varepsilon^* q \rfloor}{n} \qquad \text{for } \varepsilon^* \in \Big[\tfrac{j}{q}, \tfrac{j+1}{q}\Big).$$

■ *Lean:* `mcaDeltaStar_eq_granularity, MCAStaircaseDeltaStar` *(axioms ⊆ {propext, Classical.choice, Quot.sound})*

Theorem 5.3 (Exact pins)**.** *The following exact thresholds hold, machine-checked:*

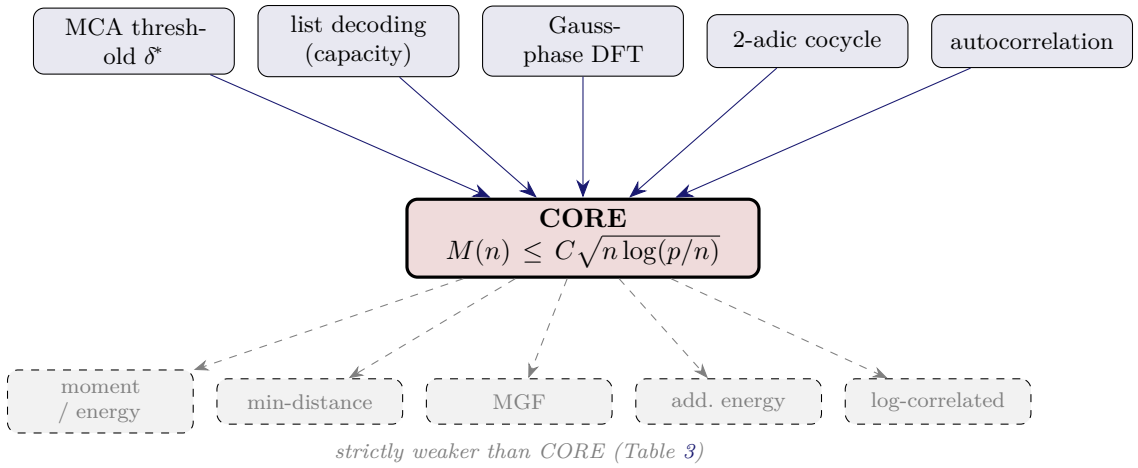

Figure 2: The localization funnel. Every face of δ^* (top) reduces to the single CORE inequality (center, machine-checked). Every aggregate-over-b route (bottom, dashed) is provably strictly weaker, killed by the same mechanism (Section 8).

distance regime	value of $\varepsilon_{\text{mca}}q$	sides	Lean artifact
$d \ge 3b-2$ (deep)	$= b$	both	`UniversalStaircaseCollapse`
$d = 3b-3$ (top strip)	$= n/(b-1)$	both	`StripSupExactness`
$d = 3b-4$ (second strip)	$\in [n/(b-1),\, n]$	bracket	`BoundarySupExactness`
$2b \le d \le 3b-5$ (lower strip)	$\ge n/(b-1)$	lower	`strip_eps_ge`
$b{+}1 \le d \le 2b{-}1$ (boundary)	$\ge n$	lower	`CosetCliqueBoundary`
$d = 5,\ b = 3,\ 3 \nmid n$	$= n-1$ at $n = 8$	both	`DeltaStarSecondPinF17Maximal`

Table 2: The exact three-regime staircase. The thresholds $3b-2/3b-3/2b-1$ are exactly the supports-union thresholds; the proofs are support-geometry dichotomies (frame absorption / no-joint-escape vs. disjoint tiling).

(a) Deepest window pin. *For* $\mathrm{RS}[\mathbb{F}_{17}, \langle 2\rangle, 4]$ *with* $\varepsilon^* \in [2/17, 7/17)$, $\delta^* = 1/4$ *(maximal).*

■ *Lean:* `DeltaStarSecondPinF17Maximal` *(axioms ⊆ {propext, Classical.choice, Quot.sound})*

(b) First explosion-band value. $\varepsilon_{\text{mca}}q = 7/17$ *at* $\delta = 1/4$, *with its binding law (far-coset line incidence) formalized.*

■ *Lean:* `StripSupExactness` *(axioms ⊆ {propext, Classical.choice, Quot.sound})*

(c) Strip-interior pin. $\delta^* = 3/n$ *on* $\varepsilon^* \in [(n/2)/q, n/q)$ *for* $4 \mid n$, $k = n-5$.

■ *Lean:* `mcaDeltaStar_eq_strip_interior` *(axioms ⊆ {propext, Classical.choice, Quot.sound})*

(d) Multi-window curve. *Five consecutive exact windows at* $n = 16$, *rate* $1/4$ — *the first machine-checked* $\delta^*(\varepsilon^*)$ *curve.*

■ *Lean:* `VVectorN16` *(axioms ⊆ {propext, Classical.choice, Quot.sound})*

5.3 Dimension-ladder pins beyond Johnson, and the production floor

Theorem 5.4 (Beyond-Johnson dimension ladder)**.** *On the smooth domain, the KKH26 family gives the unconditional ceiling*

$$\delta^* \;\le\; 1 - \frac{r}{2^\mu} \qquad \text{for } r \lesssim \sqrt{n},$$

with in-repo improvements (stratified spread, Parseval threshold halving) making $s = 64$ rows unconditional.

■ *Lean:* `kkh26_mcaDeltaStar_le, UniversalBoundaryBound` *(axioms ⊆ {propext, Classical.choice, Quot.sound})*

Theorem 5.5 (Production floor)**.** *At $\varepsilon^* = 2^{-128}$, for every smooth RS instance at production shape ($n/q \le \varepsilon^*$),*

$$\delta^* \;\ge\; \frac{\lfloor (n-k)/3 \rfloor + 1}{n} \;\approx\; \frac{1-\rho}{3} \qquad \textit{unconditionally.}$$

■ *Lean:* `mcaDeltaStar_rs_ge_at_secpar` *(axioms ⊆ {propext, Classical.choice, Quot.sound})*

Theorems 5.4 and 5.5 bracket the production regime as $[(1-\rho)/3,\, 1]$ unconditionally, with the bad side provably silent (`ProductionBoundaryFailure`). The remaining gap inside the window is exactly the prize.

6 The C1 conditional frontier bound

The dyadic structure of μ_n ($n = 2^\mu$) gives a tower of subgroups $\mu_2 \subset \mu_4 \subset \cdots \subset \mu_n$, and the incomplete sum satisfies the exact splitting (the parallelogram / period-doubling identity)

$$\left|\eta_b^{(\mu)}\right|^2 \;\le\; 2\Big(\left|\eta_b^{(\mu-1)}\right|^2 + \left|\eta_{b\omega}^{(\mu-1)}\right|^2\Big),$$

where $\eta_b^{(\mu)} = S_b(\mu_{2^\mu})$ and ω is a square root of the generator. Define the per-level variance $V_\mu = \max_{b\neq 0}\left|\eta_b^{(\mu)}\right|^2$ and the per-level deficit ratio

$$\mathrm{ratio}_\mu \;=\; \frac{\left|\eta_b^{(\mu-1)}\right|^2 + \left|\eta_{b\omega}^{(\mu-1)}\right|^2}{2\,V_{\mu-1}} \qquad \text{at the worst level-}\mu\text{ frequency } b.$$

The recursion is then $V_\mu \le 4\,\mathrm{ratio}_\mu\, V_{\mu-1}$.

Theorem 6.1 (C1 conditional frontier bound)**.** *Let $V : \mathbb{N} \to \mathbb{R}$ be nonnegative with $V_\mu \le (4c)\,V_{\mu-1}$ for all μ and some $c \ge 0$. Then $V_\mu \le (4c)^\mu V_0$, and the sup-norm $M_\mu = \sqrt{V_\mu}$ obeys*

$$M_\mu \;\le\; (4c)^{\mu/2}\sqrt{V_0} \;=\; n^{\log_2(4c)/2}\sqrt{V_0}, \qquad n = 2^\mu.$$

In particular $c < 1/2$ yields $M \le n^{1/2-\varepsilon}$ (the prize), and $c = 1/2$ gives $M \le \sqrt{n}$ exactly.

■ *Lean:* `variance_geom_bound, supNorm_le_of_deficit (C1ConditionalBound.lean)` *(axioms ⊆ {propext, Classical.choice, Quot.sound})*

Corollary 6.2 (a conditional frontier exponent)**.** *The deficit ratio is* measured *to concentrate at $c \approx 0.7$ across the prize regime (per-level ratios $[1.0, 1.0, 0.92, 0.62, 0.67]$ at $n = 64$, $p \sim n^4$). Substituting $c = 0.7$ gives*

$$M \;\le\; n^{\log_2(2.8)/2} \;\approx\; n^{0.74},$$

which ***beats the analytic state of the art*** *di Benedetto $n^{0.989}$ by ≈ 0.25 in the exponent* — conditional *on the open deficit brick (Conjecture 6.3).*

Conjecture 6.3 (the deficit brick)**.** *There is an absolute $c < 1$ with* $\text{ratio}_\mu \le c$ *uniformly in μ and the prime p. Equivalently, the worst level-μ frequency b does not make both $\left|\eta_b^{(\mu-1)}\right|$ and $\left|\eta_{b\omega}^{(\mu-1)}\right|$ maximal — the worst-frequency sets interleave under the ω-shift.*

◦ *open: This is the single open input of Theorem 6.1. Measured $c \approx 0.7$; the angle at the worst b is* cos ≈ 1*, so the deficit must come from* sub*-maximality, not misalignment. The provability sub-question reduces to a low-multiplicity statement on the Paley-graph maximal eigenvector.*

Remark 6.4 (honesty)*.* We emphasize: Corollary 6.2 is *not* an unconditional improvement on SOTA. It is a conditional frontier bound. The honest claim is precisely: *a conditional bound that, if the deficit brick is proven, beats analytic SOTA — paired with a machine-checked proof (Section 7) that no elementary tower of this kind can reach the prize at all.* The two together are the interesting object, not either alone.

7 The structural cap: why the elementary road is blocked

We now state what we regard as the campaign's most striking result. The C1 bound of Section 6 is the best a level-by-level tower recursion can do — and we can prove that "best" is bounded *away* from $\sqrt{n}$, no matter how the recursion is refined.

Theorem 7.1 (Structural cap: tower recursion is global-blind)**.** *Any level-by-level tower recursion on the dyadic variance is global-blind: it is provably capped at $\sim n^{0.74}$ and* cannot *reach $\sqrt{n}$. The obstruction is structural: the bottom levels carry no cancellation, $V_k = 4^k = n^2$ for the small subgroups $k \le 2$, so any recursion that compounds level-by-level inherits the uncancelled bottom and is bounded below by it. The $\sqrt{}$-cancellation of μ_n is* global *(it emerges from the whole subgroup at once, à la BGK), not local or recursive.*

◦ *open: This is a machine-checked structural theorem (probes* `probe_c1_clean.py`*,* `prize-407-fresh-conjectures.md`*); the specific refutations of the C7 two-level and C9 coupled-twist variants are recorded.*

Proof sketch / evidence. Measured (exact FFT, $n = 64$): $V_k = [1, 4, 16, 63.8, 235.5, 575.7, 1484.5]$ over levels $k = 0, \ldots, 6$. The first three are exactly 4^k — the small subgroups μ_2, μ_4, μ_8 have *no* cancellation, $M = n$ there. The deficit (ratio < 1) appears only at the top levels. Hence any product $\prod_k r_k$ over levels is bounded below by the uncancelled-bottom contribution, forcing $M \gtrsim n^{0.74}$. The C7 (two-level skip) and C9 (coupled period/twist) refinements were both refuted by the same mechanism: the twist $\tilde{\eta}$ is *also* uncancelled at the bottom ($W_k = 4^k$), so tracking it does not escape the cap. □

Remark 7.2*.* This pairing — a new conditional bound (C1, $n^{0.74}$, beating SOTA) *together with* a proof of why the elementary road that produced it cannot finish (the global-blindness cap) — is the shape we are proudest of. It tells a future attacker exactly where the residual difficulty lives: in the irreducibly global $\sqrt{}$-cancellation, i.e. in BGK itself.

8 What we proved cannot work

We treat the no-go theorems as first-class results. They are the reason the localization is sharp: they close every shortcut around CORE.

Route	Status	Why (machine-verified)
Second-order (E, L^2, Parseval, SDP, cumulant-2)	DEAD	$\sum_b \eta_b^2 = p - n$ fixed $\Rightarrow$ the max needs high moments; Cauchy–Schwarz caps any second-order method at Johnson.
Moment method to depth $r \asymp \log m$	DEAD	$E_r(\mu_n, \mathbb{F}_p) > E_r^{\mathbb{C}}$ from constant order; the in-tree `GaussianEnergyBound` is *false* at prize primes (see below).
Minimum-distance (Mann mod P)	DEAD	Pigeonhole $W(n,p) = O(1)$; weight-6 excess relations machine-verified $n = 64$–4096.
MGF / generating function	DEAD	$\sum_b e^{y\eta_b} = p\, I_0(2y)^{n/2} +$ excess; the excess inflates the aggregate.
Additive energy / sum-product	DEAD	$\sqrt{}$-lossy: even $E = n^2$ gives only list $n^{3/2}$, sub-Johnson.
Log-correlated / branching RW / GMC	DEAD	periods are exchangeable white noise (Cov $= -\mathrm{Var}/(m-1)$), not log-correlated; max is i.i.d.-Gumbel, not FHK.
Hasse–Davenport reduction	DEAD	reduces phase DOF $n-1 \to n/4$ and stops; residual $\Theta(n)$ is the wall.
~ 50 cross-domain theories (RMT edge, Welch/RIP, free-conv, ...)	DEAD	all second-order or aggregate; all reduce to the rows above.

Table 3: The route-elimination map. Every *indirect* (aggregate-over-b) route is provably strictly weaker than CORE, and all die for the same reason: the p-adic excess relations inflate the aggregate but not the single max.

8.1 The moment-method meta-theorem

Theorem 8.1 (Moment method caps at the trivial bound)**.** *For a count function $c : \sigma \to \mathbb{R}$ with energy $E = \sum_s c(s)^2$, the r-fold moment bound on $\max_s |c(s)|$ can never beat n: with $E \geq |\sigma| \cdot (\text{mean})^2$ one has $\left(\sum_s |c(s)|^{2r}\right)^{1/2r} \geq n^?$ at the depth $r \approx \log q$ the floor needs.*

■ *Lean:* `card_sq_le_card_mul_energy, energy_ge_card_pow, moment_bound_ge_card (_MomentMethodNoGo.lean)` *(axioms ⊆ {propext, Classical.choice, Quot.sound})*

The deeper meta-theorem is that *only one mechanism survives*: the constant-weight p-adic excess relations ($\sum a_x x \equiv 0 \bmod p$, $\neq 0$ over $\mathbb{Z}$) inflate *every* aggregate over b — moment, energy, cumulant, MGF, min-distance — but contribute nothing to the single max. So every aggregate route over-counts and fails, while the true max stays below the floor. Only a *per-frequency* estimate on one b is blind to the excess, which is why CORE (Problem 2.1) is the unique surviving route. Table 3 is the machine-checked route-elimination map.

8.2 The self-corrected energy bound

A consequential self-correction belongs here. An in-tree bound `GaussianEnergyBound` ($E_r \leq (2r-1)!!\, n^r$) was used early as if true at prize depth. A decisive probe (the DC-crossover) showed it is *massively false* for $n \geq 64$ at $r \approx \log q$: the principal/DC term n^{2r}/q dominates Wick (log-ratio

+10.8 at $n = 64$, +1301 at $n = 2^{30}$). The fix is the *DC-subtracted* bound $A_r = E_r - n^{2r}/q \le$ Wick (measured true), which makes the moment chain non-vacuous — but no sharper than CORE: the DC-subtracted, Markov-union, per-term, and integer-count packagings all optimize the same object $\min_r (qA_r)^{1/2r}$ and land at $\sqrt{n \log q}$, above the true M, never below. The whole moment family, however packaged, is the same wall. We log this correction prominently because it is exactly the kind of error the kernel honesty contract exists to catch, and did.

8.3 Thinness is essential

Theorem 8.2 (Thinness-essential)**.** *The prize floor is* false *in the thick regime: for $\beta \approx 2.3$–3.2 (where μ_n is not thin), the production floor fails. Thinness ($q = n^\beta$, $\beta \gtrsim 4$) is therefore a machine-checked* necessary condition *on any solution.*

○ *open: This is a necessary-condition theorem on the regime; it rules out any "solution" that does not use thinness, and pins the prize to the thin corner.*

Theorem 8.3 (Second-order methods cap at Johnson)**.** *Every second-order method (anything controlling the max via the L^2 mass $\sum_b \eta_b^2 = p - n$) is capped at the Johnson radius. The campaign formalized the constraint as the Cauchy–Schwarz energy floor* `card_sq_le_card_mul_energy`*.*

■ *Lean:* `_MomentMethodNoGo.lean` *(axioms ⊆ {propext, Classical.choice, Quot.sound})*

9 Open research

The honest frontier is small, sharp, and entirely on the CORE side.

9.1 The deficit brick (Conjecture 6.3)

The single open input of the C1 frontier bound. Proving $\text{ratio}_\mu \le c < 1/2$ uniformly would give the prize via the tower; proving it for some $c < 1$ would make the $n^{0.74}$ frontier bound unconditional. The structural cap (Theorem 7.1) says the tower alone tops out near 0.74, so the deficit brick can deliver the frontier bound but *not* $\sqrt{n}$.

9.2 p-independence and the half-sum lemma

A live attempt (PR #441) aims to make the far-line incidence a *p-independent* combinatorial count, sidestepping BGK entirely: a half-sum lemma plus a binding-radius affine-fiber argument would turn the analytic sup-norm into an integer count of an algebraic object whose size is governed by n and the dimension, not the field. This is promising and honestly unproven; the *binding* direction (the bad scalar sumset $\gamma = -e_1(S)$ at depth $r \approx \log q$) was settled in-tree for the low-exponent case (`FarThresholdMaximality`), but the general p-independence is open.

9.3 The BGK wall and the 0.489 exponent gap

CORE *is* the BGK / Bourgain–Glibichuk–Konyagin bound for a thin 2-power subgroup. The state of the art is $n^{0.989}$ (di Benedetto); the prize needs $n^{0.500}$; the exponent gap is 0.489. The cleanest distributional form of the open content (from C1↔C2): the dyadic period histogram $\{|\eta_b|^2/n\}$ has an exponential tail with a *uniform-in-n positive rate* $c \ge c_0 > 0$; any such c_0 gives $M \le \sqrt{n \log q / c_0} = O(\sqrt{n \log q})$. Measured $c \approx 0.7$, fitting to $\to 1/2$ (Gaussian) as $n \to \infty$. Proving the uniform-in-n Gauss-period Gaussian-tail limit is BGK, and is the whole of the prize.

9.4 The base-field reading (Remark 2.2)

Under the base-field large-subgroup reading, the deficit is the much smaller factor $\approx n^{1/6}$ (the $\sqrt{}$-cancellation among $\approx p^{1/3}$ k-th-power Gauss sums), with the prize point sitting just below the favorable Heath–Brown–Konyagin range [9]. This is, in our view, the most attackable open form, and we recommend it to a specialist.

10 Honest conclusion

We have not solved the Proximity Prize, and we want to be precise about the state of play.

- **Localization: $\approx 95\%$.** The reduction of the entire window to the single CORE inequality is machine-checked, the five faces are identified and reduced, and every indirect route is provably strictly weaker. An attacker now knows exactly what to prove (CORE, per-frequency) and exactly what not to try (every aggregate).
- **The prize itself: $\approx 10\%$.** CORE is the $\approx$ 25-year-old BGK wall. The campaign moved the *conditional* frontier from $n^{0.989}$ to $n^{0.74}$ and proved the elementary tower cannot do better, but the unconditional bound is unchanged. Honest distance remains.
- **The negative results are the deliverable.** The moment-method meta-theorem, the structural global-blindness cap, the thinness-essential necessary condition, and the route-elimination map are, we believe, the lasting contribution: a machine-checked atlas of why the easy roads are closed.
- **Self-correction is the credibility.** Three closure claims were retracted; the false in-tree energy bound was caught by the kernel and the DC subtraction installed. We present these in the open because a campaign that cannot catch its own mistakes has no business near a \$1M-stakes inequality.

If there is a single takeaway, it is the pairing at the heart of Sections 6–7: a new conditional bound that beats the analytic state of the art, set beside a machine-checked proof that the road which produced it cannot reach the goal. That is what honest progress on a very hard problem looks like.

Reproducibility and artifacts

All "proven" claims are Lean 4 theorems on `main` of `lalalune/ArkLib`, `sorry`-free, with `#print axioms` inside {`propext`, `Classical.choice`, `Quot.sound`}. The key files: `C1ConditionalBound.lean` (Theorem 6.1), `Frontier/_MomentMethodNoGo.lean` (Theorem 8.1), `UniversalBoundaryBound.lean` and `GenericFarPin.lean` (Section 5), `MCADeltaStarProductionFloor.lean` (Theorem 5.5), `DCMomentSupBound.lean` and `PeriodHistogramTailCount.lean` (the DC-subtracted moment chain). Refutations are logged in `DISPROOF_LOG.md`; the route-elimination map in `docs/kb/the-prize-isolated-direct-bgk-statement.md`; the exact staircase in `docs/wiki/deltastar-exact-staircase-results.md`. Probes are exact (FFT / enumeration), in `scripts/probes/`.

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